INSTABILITY OF AN MHD SHOCK FRONT IN THE NONLINEAR REGIME

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In this paper we study the evolution of random distortions of the front of an MHD shock wave, propagating in the direction of decreasing density of the conducting medium. It is found that a magnetic field oriented perpendicular to this direction amplifies the instability.

Initially sinusoidal distortions of the front acquire with time a more complicated form, which corresponds to sinusoidal acceleration of the motion of the advanced elements of the front and their steepening. This apparently leads to the phenomenon of toppling and turbulization of the front.

The stability of a shock-wave front with respect to its curvature has now been studied for approximately 20 years (see, for example, [1]). It has been found that the front of a shock wave in the so-called evolutionary propagation regime in a uniform medium is stable and regenerates its form free of distortions by random perturbations. However, in the case of propagation in a medium with decreasing density, both plane and spherical fronts are unstable. This was first shown for gasdynamic shock waves in the absence of a magnetic field [2, 3]. Qualitatively, the instability effect is associated with the fact that shock fronts are accelerated in a direction opposite to the density gradient. Therefore, for example, an element of the front which has accidentally moved ahead of the regular front is located in a region with lower density and its velocity is higher, so that its lead increases.

The velocity of the front as a function of its position can be successfully approximated by the formula [4] $u(x) \sim \rho^{-\lambda}$, where $\lambda = 2 + (2\gamma/\gamma - 1)^{1/2}$ and γ is the adiabatic index. A magnetic field in a conducting medium, if its intensity vector is parallel to the front, increases the effective adiabatic index of the medium [5] and correspondingly changes the parameter λ. The acceleration of the unperturbed front increases and the rate of growth of fluctuation displacements in it as a function of time also increases [6]. In addition, the latter increase with time according to a power law, but always much more rapidly than the growth of the velocity of the regular front. Of course, the existence of an instability in the theoretical respect must be confirmed by a nonlinear (with respect to the degree of deviation) calculation, which was done in [7]. Here, this calculation is performed in the general case of an MHD shock wave.

We shall study a strong MHD shock wave, in which the gas pressure in front of the front can be neglected. We shall assume that the magnetic field in the unperturbed medium $H(x)$ is oriented along the front, parallel to the y axis. In addition, we assume that the medium is ideally conducting, so that the condition of freezing in of the magnetic field is satisfied.

We assume that the curvature of the front depends on y, and we assume that the unperturbed wave propagates along the x axis. We denote the coordinate of the unperturbed front by $X(t)$ and the coordinate of the perturbed front by $\Xi(y, t) = X(t) + \xi(y, t)$ and we assume that the derivative $|d\xi/dy|$ is small enought so that the method of iterations can be used.

We shall write down the boundary conditions which relate the gasdynamic quantities in front of and behind each section of the curved front. According to [8], the local boundary conditions, corresponding to continuity of the flows of mass, momentum, energy, magnetic-field component normal to the section of the front, and of the tangential component of the electric field have the following form:

$$\{\rho(v_x - \dot{\Xi}\cos^2\theta)\} = 0,$$

$$\left\{-\rho\dot{\Xi}^2\cos^2\theta + p + \rho v_x^2 + \frac{H_y^2 - H_x^2}{8\pi} - \operatorname{tg}\theta\left(\rho v_x v_y - \frac{H_x H_y}{4\pi}\right)\right\} = 0,$$

$$\left\{-\rho v_y\dot{\Xi} + \left(\rho v_x v_y - \frac{H_x H_y}{4\pi}\right) - \operatorname{tg}\theta\left(p + \rho v_y^2 + \frac{H_x^2 - H_y^2}{8\pi}\right)\right\} = 0,$$

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$$\Big\{-\dot{\Xi}\left(\frac{\rho v^2}{2}+\rho\varepsilon+\frac{H^2}{8\pi}\right)+\rho v_x\left(\frac{v^2}{2}+w\right)+\frac{1}{4\pi}\left(v_x H^2-H_x \mathbf{v}\cdot\mathbf{H}\right)-$$
$$-\operatorname{tg}\theta\rho v_y\left(\frac{v^2}{2}+w\right)-\frac{1}{4\pi}\operatorname{tg}\theta\left(v_y H^2-H_y \mathbf{v}\cdot\mathbf{H}\right)\Big\}=0,$$
$$\{H_x-\operatorname{tg}\theta\, H_y\}=0,\ \{(v_x-\dot{\Xi})H_y-v_y H_x\}=0.$$

Here, the braces indicate the difference in the corresponding quantities behind and in front of the section of the front; p, ρ, ε, w are the pressure, density, specific internal energy, and enthalpy of the gas behind the front; the zero index denotes unperturbed quantities in front of the front; v_x and v_y are the components of the gas velocity in the laboratory coordinate system, i.e., in the coordinate system where the gas is at rest in front of the front; $\theta = \theta(y, t)$ is the slope angle of the section of the curved front in the (y, z) plane of the unperturbed front, so that $\cos\theta = 1 - (1/2)(d\xi/dy)^2$. In the absence of a magnetic field, the boundary conditions assume the form of the well-known relations on an inclined shock front [7].

Eliminating the quantities v_y, H_x, we obtain a system of three independent equations, determining the shock adiabat:

$$\{\rho(v_x-\dot{\Xi}\cos^2\theta)\}=0, \tag{1}$$
$$\left\{-\rho\dot{\Xi}^2\cos^2\theta+p^*+\frac{\rho v^2}{\cos^2\theta}+\frac{\operatorname{tg}^2\theta}{4\pi}H_0\left(H_y-H_0\right)\right\}=0,$$
$$\left\{\rho\left(v_x-\dot{\Xi}\right)\left(\frac{v^2}{2}+w^*\right)+p^*\dot{\Xi}+\operatorname{tg}^2\theta\left[\rho v_x\left(\frac{v^2}{2}+w^*-\frac{H_0^2}{4\pi\rho}\right)\right]\right\}=0,$$

where $p^* = p + H^2/8\pi$; $w^* = w + H^2/4\pi\rho$; $p_0^* = H_0^2/8\pi$. If the front of the shock wave were to be displaced from the point X to the point $(X + \xi(t))$, while remaining in the (y, z) plane, then its velocity would be equal to the value $u(X+\xi)$, i.e., the function determined by the solution of the problem of the motion of the unperturbed front in a medium with a nonuniform density. We shall assume that this solution (for example, of a self-similar type) is known. In this case, the boundary conditions on the front are satisfied for some definite values of the gasdynamic quantities $p=\tilde{p}(X+\xi)$, $\rho=\tilde{\rho}(X+\xi)$, $v=\tilde{v}(X+\xi)$, $H=H(X+\xi)$. Small distortions of the front also cause, generally speaking, small changes in the gasdynamic quantities. In the boundary conditions (1), we now write $p=\tilde{p}+\delta p$, $\rho=\tilde{\rho}+\delta\rho$, $\dot{\Xi}=u(X+\xi)+\delta u$, where $\delta u=\dot{\xi}-\xi du/dX-(1/2)\xi^2 d^2u/dX^2+\ldots$.

We shall now implement the iteration procedure. Thus, in the first approximation with respect to the curvature parameter of the front $|d\xi/dy| \ll 1$, we obtain a system of equations for the indicated small deviations of the gasdynamic quantities, which in the case of a perfect gas with an adiabatic index according to [6] has the form

$$\left(\frac{u}{v}-1\right)\frac{\delta\rho_1}{\rho}-\frac{\delta v_1}{v}+\frac{\delta u_1}{u}=0, \tag{2}$$
$$\left(1-\frac{\rho c_s^*}{\rho_0 u}\right)^{-1}\frac{\delta u_1}{u}+\frac{\delta v_1}{v}=0,$$
$$\left(\frac{H_0^2\rho}{8\pi\rho_0^2}-\frac{p}{\rho(\gamma-1)}\right)\frac{\delta\rho_1}{\rho}+\left(\frac{c_s^2 v}{(\gamma-1)c_s^*}-v^2-\frac{p_0^* v}{\rho_0 u}\right)\frac{\delta v_1}{v}+\frac{p_0^* v}{\rho_0 u}\frac{\delta u_1}{u}=0,$$

where the index 1 denotes fluctuations of first order with respect to the curvature parameter $|d\xi/dy|$. In addition, the following relations are used: $\delta p^*=-\rho c_s^*\delta v$, $\delta p=-(\rho c_s^2/c_s^*)\delta v$, $c_s^2=\gamma p/\rho$, $c_s^*=\sqrt{\gamma p/\rho+H^2/8\pi}$. The last relations occur in the so-called quasiclassical approximation, when the inequality $kl \gg 1$, where $k = k_y$ is the characteristic wave number of the perturbation and $l = (d\ln\rho_0/dx)^{-1}$ is the characteristic distance over which the density changes, holds [6, 7]. In order to resolve the question of instability, it is sufficient to determine if there are any perturbations which grow with time. In preceding studies it was shown that

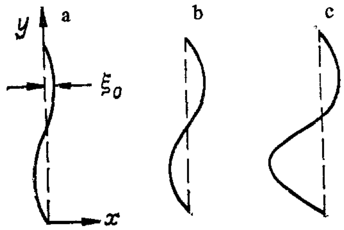

Fig. 1

the form of the perturbations studied here, corresponding to the quasiclassical approximation, leads to an instability of the front. Over a wide range of variation of the adiabatic index $1 < \gamma < 2$, the solution of the system (2) has the form $\delta p_1^* = 0$, $\delta\rho_1 = 0$, $\delta u_1 = 0$. From here follows the spontaneous growth of the displacement ξ (for either sign of this quantity), because the quantity du/dX is positive for a very wide class of solved problems.

We shall examine fluctuations of gas-dynamic quantities on the front in second order with respect to the curvature parameter. In this case, because $\delta p_1^* = 0$, $\delta\rho_1 = 0$ and so on, we can write $\delta p = \delta p_2,\ \delta v = \delta v_2,\ \delta u = \delta u_2 = \dot{\xi}_2 - \xi_2 \frac{du}{dX} - \frac{\xi_1^2}{2}\frac{d^2u}{dX^2}$. Iteration in the system (1) in the approximation studied leads to the following inhomogeneous system of equations with respect to the enumerated corrections:

$$-\rho_0 u\frac{\delta\rho}{\rho} + \rho\delta v - (\rho - \rho_0)\,\delta u = -\theta^2(\rho - \rho_0)\,u,$$

$$\frac{\rho}{\rho_0}u_A^2\delta\rho + \left(\pm\rho c_s^* - \rho_0 u\right)\delta v - \rho_0 v\delta u = -\theta^2\left(\rho v^2 + \frac{H_0}{4\pi}(H_y - H_0)\right),$$

$$\frac{u_A^2}{\rho_0}\delta\rho + \left(v - u \pm c_s^*\right)\delta v - v\delta u = \frac{\theta^2}{\rho_0 u}\left(\rho u v^2 + \frac{H_0}{4\pi}u\,(H_y - H_0) - \rho v\left(\frac{v^2}{2} + w^*\right)\right).$$

We obtain a solution of this system under the assumption that the parameter $h^2 = H_0^2/8\pi p$ is small, i.e., taking into account the magnetic pressure only as a correction. For $h^2 = 0$, the solution of the system is presented in [7]. Here, we take into account the corrections arising due to the magnetic pressure, in order to determine the effect of the magnetic field at the nonlinear stage of development of the instability of the front. It turns out that the displacements $\xi_{1,2}$, in the first and second approximations, respectively, are coupled by the following differential equation:

$$\frac{d\xi_2}{dx} - \xi_2\frac{d\ln u}{dx} = \frac{\xi_0 u}{2u_0^2}\frac{d^2u}{dx^2} + \theta^2(1 - \eta f(\gamma)).$$

For the case of an exponential dependence of the density $\rho_0 \sim \exp(-x/l)$

$$\xi_2 = \left(\frac{\lambda\xi_0^2}{2l} + \frac{1-\eta f}{\lambda}\theta_0^2\right)(l^{2z} - l^z),$$

$$z \equiv \frac{\lambda X(t)}{l},\quad \theta_0 = \theta(t = 0).$$

Here

$$f(\gamma) = \frac{(\gamma+1)\gamma}{2} - \frac{\gamma}{\gamma+1} - \frac{(\gamma^2-1)(\gamma-1)}{4} - \frac{(\gamma+1)\lambda}{\gamma-1},\quad \eta = \frac{4h^2}{(\gamma-1)^2},\tag{3}$$

$$X(t) = X_0 + \frac{l}{\lambda}\ln\frac{\lambda u t}{l},$$

and, in addition, $\xi_2 < \xi_1$, if $\lambda\xi_0 < l$ (this inequality is satisfied when $\xi_0 < l$).

Figure 1 shows the numerically computed change in the form of the distortions on the front as the unperturbed front moves forward. The position of the latter for successive values of the dimensionless coordinate z is marked by the line. The starting distortion has a sinusoidal form $\xi = \xi_0 \sin \varkappa y$, and the following values of the parameters are adopted: $\gamma = 5/3$, $h^2 = 1/9$, $\theta_0 = 0.1$, $\xi_0 = 0.1l$ ($\varkappa\xi_0 = \theta_0$). Figure 1a-c refers to the positions $z = 0$, 1, and 1.5 and the corresponding times are determined by the dependence $X(t)$ in Eq. (3). The extrapolation of the results to large z can only have a qualitative meaning, because higher-order nonlinearities enter the picture. In the example studied, the rate of development of the instability is approximately two times higher than in the absence of a field.

If in the first approximation the growth in the displacements of the front from its equilibrium position occurs symmetrically relative to the lagging and leading sections, then asymmetry appears when nonlinear corrections are included: Leading occurs relatively more rapidly than lagging of the elements of the front, and this effect is all the more significant the higher the amplitude of the displacements and the curvature of the front. The correction arising due to the magnetic pressure, as calculations with all adiabatic indices in the interval indicated show, has one and the same sign. For small enough h, this increases the slope of the front, because its protrusions with $\theta \neq 0$ are accelerated more strongly with respect to the side sections than in the absence of a magnetic field. In this case, $\gamma = 2; 5/3; 4/3; 6/5$ and $f = -0.08; -1.13; -5.19; -26.22$.

Thus a magnetic field always amplifies the instability, because it increases the rate of growth of the velocity of the unperturbed front with time. We note that the instability which we studied does not reduce to known forms of instability of the motion of a conducting medium, for example, to a Rayleigh–Taylor instability. Indeed, in a system in which the unperturbed front is at rest, the less dense unperturbed medium is accelerated toward the front so that a light gas is more effective than a heavy gas, and vice versa.

The instability examined can play a definite role in the motion of shock waves in a cosmic plasma and, apparently, in the laboratory plasma also [9], for example, in setups in which the pinch effect appears.

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